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Field redefinition rules for auxiliary field searches

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Abstract. We show how to redefine differentially constrained and unconstrained fields so that they become either auxiliary or propagating fields when the equations of motion are used.

1. Introduction

Massless off-shell N-extended supersymmetry and supergravity is expected to be satisfactorily analysed, at the linearised level, in terms of massive irreducible representations (irreps) of the appropriate supersymmetry algebra S_N. These and their symmetries are now known (Taylor 1980, 1981b, 1982a, Pickup and Taylor 1981, Bufton and Taylor 1982b, Ferrara *et al* 1980, Sokatchev 1982) for all physically interesting values of N, so that it is to be expected that this knowledge would have had immediate application in effortlessly constructing the corresponding off-shell linearised theories. That this has not happened is due to the SU(N) symmetry present in S_N for $N \geqslant 3$ (Rivelles and Taylor 1981, Taylor 1982b), but a contributing feature making the analysis difficult is that the component fields in irreps of S_N are not the gauge-dependent physical fields whose on-shell restrictions describe only the physically desired helicity states. Crucial field recombinations have to be taken before the usual gauge fields of supersymmetry and more particularly of supergravity emerge. In the process, different irreps are mixed together in a decidedly non-trivial fashion.

The purpose of this paper is to develop to the full such field redefinitions as presently appear of value, especially in constructing off-shell supergravities. Since these redefinition rules have already given auxiliary field candidates for linearised $N \geqslant 3$ extended supergravities (Taylor 1981a) and new sets of auxiliary fields for $N = 1$ and 2 linearised supergravities (Rivelles and Taylor 1982b, c, d) the detailed tabulation and analysis of such rules would seem timely.

Field redefinition rules appear first to have been used, somewhat implicitly, in a search for the auxiliary fields of $N = 2$ supergravity (de Wit and van Holten 1979). They were then made explicit and used (Rivelles and Taylor 1982a) in an analysis of the superfield structure of linearised $N = 2$ supergravity. Since their value is only apparent when the irrep structure of S_N is known and these latter have only been recently obtained, their late appearance as a useful research tool in determining off-shell structure specifically in terms of auxiliary fields becomes understood.

We wish to distinguish between two sorts of field redefinition rules for their application to supergravity. The first class we call annihilation rules and it allows two

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or more otherwise propagating fields (sometimes with negative kinetic energy) to be combined together so that when the equations of motion are used, the resulting field vanishes or only its gauge modes survive. Such fields are usually called auxiliary fields. The other class of redefinition rules are called creation rules. It is these rules which combine fields of various spin together so that an appropriate unconstrained field is obtained which has a certain amount of gauge invariance in its Lagrangian. It is this latter which allows the combined field to describe a massless field of unique spin. The epithet 'creative' is thus appropriate here since the resulting field is usually regarded as the physical one, and has been created by field combinations from otherwise unphysical modes in the original irreps. The unphysical features of the latter arise either due to differential constraints or negative kinetic energy terms.

In the next section we develop the annihilation rules for fermions and extend these to bosons in the following section. Fermionic creation rules are considered next and then we discuss boson creation rules. A final section summarises our results and discusses avenues for future work.

2. Fermion annihilation rules

The most elementary of all fermionic annihilation rules, and the basis of all other known ones, is that for Majorana spin-$\frac{1}{2}$ particles. If we denote by $\pm\frac{1}{2}$ the signed kinetic energy term $\pm\bar{\psi}\not{p}\psi$, we have that

$$\bar{\psi}\not{p}\psi-\bar{\phi}\not{p}\phi=(\bar{\psi}+\bar{\phi})\not{p}(\psi-\phi)\equiv\bar{\lambda}_1\lambda_2, \tag{2.1}$$

to within a total divergence (which we neglect in the present analysis), where we have defined the Majorana spinors λ_1 and λ_2 of (mass) dimensions $\frac{3}{2}$ and $\frac{5}{2}$ as $\lambda_1=\psi+\phi$, $\lambda_2=\not{p}(\psi-\phi)$. Then (2.1) may be written as

$$\tfrac{1}{2}-\tfrac{1}{2}\approx 0\,, \tag{2.2}$$

where ≈ 0 denotes the vanishing of the combination of spin-$\frac{1}{2}$ fields on the LHS by their combined equations of motion $\lambda_1=\lambda_2=0$. The fermions λ_1 and λ_2 are therefore auxiliary fields.

We emphasise here that the interpretation of (2.2) is not in terms of the propagating fields on the LHS. The independent fields of the combination of the LHS of (2.2) are the spinors λ_1 and λ_2, in terms of which ψ and ϕ can be trivially, but non-locally, expressed as $\psi=\frac{1}{2}(\lambda_1+\not{p}^{-1}\lambda_2)$, $\phi=\frac{1}{2}(\lambda_1-\not{p}^{-1}\lambda_2)$. Thus there are no physical modes in terms of the underlying fields λ_1 and λ_2. We would be mistaken in considering (2.2) to be composed of a physical and ghost spin-$\frac{1}{2}$ pair of propagating particles; there is no physical content at all in (2.2) when considered in terms of the fields λ_1 and λ_2. We note that the problems associated with the ghost in (2.2) can only be removed satisfactorily by regarding λ_1 and λ_2 as the fundamental fields. The search for auxiliary fields is that of finding such fields which do annihilate apparently propagating combinations, as given by the LHS of (2.2). We should also remark here that though local field redefinitions cannot change dynamics, the non-local ones we are contemplating can do so, and indeed must do so in order to produce a physically reasonable result (in that ghosts are thereby removed).

The next annihilation rule involves spins $\frac{3}{2}$ and $\frac{1}{2}$. By analogy with the RHS of (2.1) we expect it to arise from differentially unconstrained vector spinors $\lambda_{1\mu}$, $\lambda_{2\mu}$ in the form $\bar{\lambda}_1^{\mu}\lambda_{2\mu}$. To obtain this term we start with the pure spin-$\frac{3}{2}$ Majorana fields ψ_μ,

ϕ_μ, for which $p^\mu\psi_\mu = p^\mu\phi_\mu = \gamma^\mu\psi_\mu = \gamma^\mu\phi_\mu = 0$. We remove both constraints on ψ_μ and ϕ_μ, by writing the pure spin-$\frac{3}{2}$ field ψ_μ in terms of the unconstrained field λ_μ as

$$\psi_\mu = \tilde{\eta}_{\mu\nu}\lambda^\nu - (1/3)\gamma'_\mu\gamma_\nu\tilde{\eta}^{\nu\lambda}\lambda_\lambda, \tag{2.3}$$

with $\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu/p^2$, $\gamma'_\mu = \gamma_\mu - p_\mu \not{p}/p^2$. Equation (2.3) thus allows the decomposition of a completely unconstrained vector spinor χ_μ into the spin-$\frac{3}{2}$ and -$\frac{1}{2}$ irreps of the Lorentz group as

$$\chi_\mu = \psi_\mu + \mathrm{i}(\gamma_\mu - 4p_\mu\not{p}^{-1})\psi + \mathrm{i}(\gamma_\mu - p_\mu\not{p}/p^2)\psi'. \tag{2.4}$$

Furthermore, $\bar{\chi}^\mu\not{p}\chi_\mu = \bar{\psi}^\mu\not{p}\psi_\mu + 6\bar{\psi}\not{p}\psi - 3\bar{\psi}'\not{p}\psi' - 6\bar{\psi}'\not{p}\psi$. We can diagonalise the last three terms by suitable linear combinations of ψ and ψ' since the eigenvalues of the quadratic form in ψ and ψ' are $\frac{3}{2}(1 \pm \sqrt{13})$. In terms of the diagonal modes ψ_1 and ψ_2 we have

$$\bar{\psi}^\mu\not{p}\psi_\mu = \bar{\chi}^\mu\not{p}\chi_\mu + \bar{\psi}_1\not{p}\psi_1 - \bar{\psi}_2\not{p}\psi_2. \tag{2.5}$$

We perform the same decomposition for ϕ_μ as

$$\bar{\phi}^\mu\not{p}\phi_\mu = \bar{\eta}^\mu\not{p}\eta_\mu + \bar{\phi}_1\not{p}\phi_1 - \bar{\phi}_2\not{p}\phi_2. \tag{2.6}$$

The recombination (2.1) now gives

$$\bar{\psi}^\mu\not{p}\psi_\mu - \bar{\phi}^\mu\not{p}\phi_\mu \equiv \bar{\lambda}_1^\mu\lambda_{2\mu} + \bar{\lambda}_1\lambda_2 + \bar{\lambda}_3\lambda_4, \tag{2.7}$$

where $\lambda_{1\mu} = \chi_\mu + \eta_\mu$, $\lambda_{2\mu} = \not{p}(\chi_\mu - \eta_\mu)$, $\lambda_1 = \psi_1 + \phi_1$, $\lambda_2 = \not{p}(\psi_1 - \phi_1)$, $\lambda_3 = \psi_2 + \phi_2$, $\lambda_4 = \not{p}(\phi_2 - \psi_2)$. Equation (2.7) gives the rule

$$\tfrac{3}{2} - \tfrac{3}{2} \approx 0. \tag{2.8}$$

We may extend (2.2) and (2.8) to higher spin straightforwardly. For higher half-odd-integer spin j the above methods lead to the annihilation rules

$$j - j \approx 0, \ (j = \text{half-odd-integer}), \tag{2.9}$$

where the RHS of (2.9) is the appropriate generalisation of the RHS of (2.7).

We may combine various spins together to give auxiliary fields of simpler form. Thus by taking the last two terms on the RHS to the LHS of (2.7), and using (2.1), we have

$$(\tfrac{3}{2} + \tfrac{1}{2}^2) - (\tfrac{3}{2} + \tfrac{1}{2}^2) \approx 0, \tag{2.10}$$

where the RHS now denotes $\bar{\lambda}_1{}^\mu\lambda_{2\mu}$. More generally (2.10) becomes

$$[j + (j-1)^2 + \ldots + \tfrac{1}{2}^2] - [j + (j-1)^2 + \ldots + \tfrac{1}{2}^2] \approx 0 \tag{2.12}$$

where the RHS denotes $\bar{\lambda}_1^{\mu_1\ldots\mu_{j-\frac{1}{2}}}\lambda_{2\mu_1\ldots\mu_{j-\frac{1}{2}}}$. We have used this version of the fermion annihilation rules elsewhere (Rivelles and Taylor 1981, Taylor 1982b).

The use of other unconstrained tensor spinors such as $\chi_{\mu\nu} = -\chi_{\nu\mu}$ only produces multiples of the rules (2.9), such as $j^2 - j^2 \approx 0$, but never rules involving an odd total number of fermions being annihilated.

3. Boson annihilation rules

One method of constructing boson annihilation rules is identical to that for fermions, i.e.: express a given spin field in terms of a differentially unconstrained tensor and lower spin fields for which annihilation rules have already been obtained. Equivalently we may decompose a differentially unconstrained tensor into its given spin components and so obtain annihilation rules involving fields of different spin. It is in this latter form that we will obtain our rules, similarly to those of (2.12). Another method is to rewrite combinations of constrained fields in terms of gauge fields restricted by the equations of motion to pure gauge modes.

Using the first method, the simplest boson rule is obtained from the unconstrained real vector A_μ, which we may decompose into its spin 0 and 1 components as

$$A_\mu = (\delta_\mu{}^\nu - p_\mu p^\nu/p^2)A_\nu + \mathrm{i}p_\mu\phi, \tag{3.1}$$

where $\phi = -\mathrm{i}p_\mu/p^2 A^\mu$. Then to within a total divergence

$$-A^\mu A_\mu = -\phi p^2\phi - B^\mu B_\mu, \tag{3.2}$$

where $B_\mu = (\eta_{\mu\nu} - p_\mu p_\nu/p^2)A^\mu$, so $p^\mu B_\mu = 0$.

The second term on the RHS of (3.2) is the Lagrangian for a real vector field B_μ of dimension 2, and we denote it by 1_A. The first term in (3.2) is the negative of the Lagrangian for the real scalar ϕ of dimension 1; we denote this by $-O_p$. Then (3.2) gives the annihilation rule

$$1_A - O_p \approx 0, \tag{3.3}$$

where the RHS of (3.3) is $A_\mu{}^2$, in terms of the auxiliary field A_μ.

For spin 2 we consider the symmetric traceless tensor $S_{\mu\nu}$ with 9 independent components. We use the decomposition

$$S_{\mu\nu} = h_{\mu\nu} + \mathrm{i}p_{(\mu}V_{\nu)} + (2p_\mu p_\nu/p^2 - 1/2\eta_{\mu\nu})S \tag{3.4}$$

in terms of the symmetric traceless tensor $h_{\mu\nu}$ with $p^\mu h_{\mu\nu} = 0$ and the constrained vector V_μ with $p^\mu V_\mu = 0$. To within a total divergence

$$S_{\mu\nu}{}^2 = h_{\mu\nu}{}^2 + F_{\mu\nu}{}^2(V) + S^2, \tag{3.5}$$

the RHS being the sum of suitably signed spins 2, 1 and 0 respectively. We can therefore rewrite (3.5) as the annihilation rule

$$2_A - 1_p + O_A \approx 0. \tag{3.6}$$

We can generalise (3.3) and (3.6) by taking the (λ, λ) representation of SO(3,1) and writing the annihilation rule

$$\eta_A - (\eta - 1)_p + (\eta - 2)_A - \ldots \approx 0, \qquad \eta = 2\lambda. \tag{3.7}$$

Annihilation rules involving 1_p (a vector of dimension 1) may be obtained from other unconstrained tensors. Thus the tensor $A_{\mu\nu}$ antisymmetric in $\mu \leftrightarrow \nu$ may be decomposed as

$$A_{\mu\nu} = \mathrm{i}p_{[\mu}A_{\nu]} + \mathrm{i}\varepsilon_{\mu\nu\lambda\sigma}p^\lambda V^\sigma, \tag{3.8}$$

and again to within total divergences

$$A_{\mu\nu}{}^2 = F_{\mu\nu}{}^2(A) - F_{\mu\nu}{}^2(V), \tag{3.9}$$

where $F_{\mu\nu}(W) = \partial_\mu W_\nu - \partial_\nu W_\mu$. Equation (3.9) can be rewritten

$$1_p - 1_p \approx 0. \tag{3.10}$$

This may be generalised to a second class of annihilation rules, obtained from the irreps $(\lambda, \mu)+(\mu, \lambda)$ of SO(3,1); the general case is

$$[(\lambda+\mu)_{P,A}+(\lambda+\mu-1)_{A,P}+\ldots\pm(\lambda-\mu)] \\ -[(\lambda+\mu)_{P,A}+(\lambda+\mu-1)_{A,P}+\ldots\pm(\lambda-\mu)]\approx 0 \qquad (\lambda>\mu) \tag{3.11}$$

where we choose P for λ odd or A for λ even in the first bracket and it is alternated in the following ones.

The second method of obtaining annihilation rules is most easily seen by considering two physical scalars A_1 and A_2:

$$A_1p^2A_1-A_2p^2A_2=B_1B_2, \tag{3.12}$$

where $B_1=A_1+A_2$ and $B_2=p^2(A_1-A_2)$ with dimensions 1 and 3 respectively. We can then write (3.12) as the annihilation rule

$$O_p-O_p\approx 0. \tag{3.13}$$

Such new types of auxiliary fields have already appeared in the new versions of $N=1$ and $N=2$ supergravity (Rivelles and Taylor 1982b, c, d). For spin 1 we have the annihilation rule

$$1_A-1_A\approx 0 \tag{3.14}$$

which has been used to construct the new minimal version of $N=1$ supergravity (Sohnius and West 1981). This rule follows from

$$A^2{}_\mu-B^2{}_\mu=U_\mu\varepsilon^{\mu\nu\lambda\sigma}\partial_\nu W_{\lambda\delta} \\ \delta U_\mu=\partial_\mu\Lambda, \qquad \delta W_{\mu\nu}=\partial_{[\mu}\Lambda_{\nu]} \tag{3.15}$$

where $U_\mu=A_\mu-B_\mu$, $W_{\mu\nu}=(1/2p^2)\varepsilon_{\mu\nu\lambda\sigma}\partial^\lambda(A^\sigma+B^\sigma)$ and $p^\mu A_\mu=p^\mu B_\mu=0$. The resulting field equations give $W_{\mu\nu}$ and U_μ as purely gauge modes. We can also take P instead of A in (3.14) by the same method.

We may generalise (3.13) and (3.14) to higher spin, as

$$j-j\approx 0 \tag{3.16}$$

where either P or A can be used to label j in (3.16).

4. Fermionic and bosonic creation rules

As we have already remarked, creation rules are those which allow the combination of differentially constrained fields of given spin to be taken to produce a propagating field without ghosts but with suitable gauge invariance to lead on-shell to a massless field of given helicity. Due to the difficulties associated with the coupling of higher spin fields we will only consider the case with spin $\frac{3}{2}$ and 2.

The appropriate gauge-invariant Lagrangian for a spin-$\frac{3}{2}$ massless Majorana field ψ_μ is that of Rarita and Schwinger

$$L_{\mathrm{RS}}=-\tfrac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu p_\lambda\psi_\sigma, \tag{4.1}$$

which is invariant under $\delta\psi_\mu=\partial_\mu\varepsilon$. The field ψ_μ has 12 degrees of freedom when the gauge invariance is taken into account, so that 4 further degrees of freedom are needed to combine with the 8 degrees of freedom of a pure spin-$\frac{3}{2}$ field ϕ_μ. We expect these extra degrees of freedom to be given by a spin-$\frac{1}{2}$ field, and thus that

$$\tfrac{3}{2}-\tfrac{1}{2}\approx L_{\mathrm{RS}}. \tag{4.2}$$

This creation rule can be proved by using the expression for a pure spin-$\frac{3}{2}$ field ϕ_μ in terms of an unconstrained vector spinor ψ_μ as (see (2.3))

$$\phi_\mu=\bar{\eta}_{\mu\nu}\psi^\nu-\tfrac{1}{3}\bar{\eta}_{\mu\lambda}\gamma^\lambda\gamma^\sigma\bar{\eta}_{\sigma\nu}\psi^\nu. \tag{4.3}$$

We may then evaluate the Lagrangian for ϕ_μ in terms of ψ_μ, and obtain

$$\bar{\phi}^\mu \not{p}\phi_\mu=-\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu p_\rho\psi_\sigma+\tfrac{2}{3}\bar{\lambda}\not{p}\lambda, \tag{4.4}$$

where $\lambda=\gamma^\mu\psi_\mu-\not{p}p^\mu/p^2\psi_\mu$. On taking the second term on the RHS of (4.4) to the LHS and using (4.1) we obtain (4.2). This is the explicitly Lorentz covariant generalisation of an earlier decomposition (Deser *et al* 1977).

For the bosonic case we use that the differentially constrained symmetric and traceless spin-2 field $h_{\mu\nu}$ can be expressed in terms of the unconstrained symmetric field $f_{\mu\nu}$ as

$$h_{\mu\nu}=\bar{\eta}_{\mu\rho}\bar{\eta}_{\nu\sigma}f^{\rho\sigma}-\tfrac{1}{3}\bar{\eta}_{\mu\nu}\bar{\eta}_{\rho\sigma}f^{\rho\sigma}. \tag{4.5}$$

Then the Lagrangian $h^{\mu\nu}p^2h_{\mu\nu}$ can be rewritten in terms of $f_{\mu\nu}$ as

$$h^{\mu\nu}p^2h_{\mu\nu}=2L_{\mathrm{E}}(f)+\tfrac{2}{3}bp^2b \tag{4.6}$$

where $L_{\mathrm{E}}(f)$ is the linearised Einstein Lagrangian

$$L_{\mathrm{E}}(f)=\tfrac{1}{2}f^{\mu\nu}p^2f_{\mu\nu}-f^{\mu\nu}p_\mu p^\rho f_{\rho\nu}+f_{\mu\nu}p^\mu p^\nu f_\rho{}^\rho-\tfrac{1}{2}f_\mu{}^\mu p^2 f_\nu{}^\nu,$$

and $b=f_\mu{}^\mu-p^\mu p^\nu/p^2f_{\mu\nu}$ being the dilation mode. We may thus write (4.6) as

$$2_p-O_p\approx L_{\mathrm{E}}, \tag{4.7}$$

being the appropriate creation rule for spin 2. This is the explicitly Lorentz covariant generalisation of the canonical decomposition in Arnowitt *et al* (1962).

5. Summary and discussion

We have obtained the rules which have helped the search for auxiliary fields in supergravity. These rules are of two sorts: annihilation and creation rules. The former allow field recombinations to be made which cause unwanted fields to become gauge modes or to vanish on-shell. They therefore provide the auxiliary fields directly. The other rules provide the determination of lower spin companions to spin-$\frac{3}{2}$ or -2 fields so that the latter are described by differentially unconstrained fields with an associated gauge invariance. These latter fields appear as the physically appropriate way to describe massless fields off-shell. In particular the lack of differential constraints on the fields allows for their satisfactory quantisation.

There are many further steps to be taken after this to lead to a fully nonlinear off-shell theory of $N=8$ supergravity. But even the use of the creation rules (4.2) and (4.7) alone indicates that more than one supersymmetric multiplet must be used, since all the component fields in a single multiplet will have the same sign of kinetic energy in a linearised Lagrangian.

One feature which requires further study is that of putting the rules on a nonlinear footing. Especially we need to determine in what way they are modified by the presence of an arbitrary curved background space–time. For example, terms dropped as total derivatives on integration by parts may now become important in topological non-trivial space–times and would change the rules.

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References

Arnowitt R, Deser S and Misner C W 1962 *Gravitation: an Introduction to Current Research* ed L Witten (New York: Wiley) p 227
Bufton G and Taylor J G 1982 *Representations of Global Supersymmetries for all N, Kings College preprint*
Deser S, Kay J H and Stelle K S 1977 *Phys. Rev.* **16** 2448
Ferrara S, Savoy C A and Zumino B 1980 *Phys. Lett.* **100**B 393
Pickup C and Taylor J G 1981 *Nucl. Phys.* B **188** 577
Rivelles V O and Taylor J G 1981 *Phys. Lett.* **104**B 131
—— 1982a *J. Phys. A: Math. Gen.* **15** 163
—— 1982b *Phys. Lett.* B to appear
—— 1982c *All Possible Sets of Lower Spin Auxiliary Fields for N = 1 Supergravity, Kings College preprint* (*February*)
—— 1982d *All Other Minimal Sets of Auxiliary Fields for N = 2 Supergravity, Kings College preprint* (*May*)
Sohnius M F and West P C 1981 *Phys. Lett.* **105B** 359
Sokatchev E 1982 *Supergravity '81* ed S Ferrara and J G Taylor (Cambridge: Cambridge University Press)
Taylor J G 1980 *Nucl. Phys.* B **169** 484
—— 1981a *Phys. Lett.* **105**B 429, 434
—— 1981b *Extended Superfields in Linearised Supersymmetry and Supergravity* ed S Hawking and M Rocek (Cambridge: Cambridge University Press)
—— 1982a *Supergravity '81* ed S Ferrara and J G Taylor (Cambridge: Cambridge University Press)
—— 1982b *J. Phys. A: Math. Gen.* **15** 867
de Wit B and van Holten J W 1979 *Nucl. Phys.* B **155** 530